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Asymptotic behavior of the energy for a class of weakly dissipative second-order systems with memory ☆

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Abstract

A class of second-order abstract systems with memory and Dirichlet boundary conditions is investigated. By suitable Liapunov functionals, existence of solutions as well as asymptotic behavior, are determined. In particular, when the memory kernel decays exponentially, the polynomially decay of the solutions is proved.

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1. Introduction

Let $\mathcal{A}$ be a strictly positive, self-adjoint operator with domain $\mathcal{D}(\mathcal{A}) \subset \mathcal{H}$ a Hilbert space. We introduce a class of second-order abstract models

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$$u_{tt}(t) + \mathcal{A}u(t) + \beta u(t) - (g * \mathcal{A}^{\alpha} u)(t) = 0 \quad \text{in } L^2(0, T; \mathcal{H}),$$
$$u(0) = u_0, \quad u_t(0) = u_1 \quad \text{in } \mathcal{H}, \tag{1}$$

where $u_0 \in \mathcal{D}(\mathcal{A})$ and $u_1 \in \mathcal{D}(\mathcal{A}^{1/2})$. Here α is a real number in the range $[0, 1]$, β is a non-negative constant, $*$ is the convolution product, the subscript t denotes time derivative. The function $g \in C^1 \cap L^1(\mathbb{R}^+)$ is the kernel associated to system (1).

The system above may represent an isotropic viscoelastic model when $\alpha = 1$, $\beta = 0$, and, for instance, $\mathcal{A}u = -\mu \Delta u - (\lambda + \mu)\nabla(\operatorname{div} u)$ or $\mathcal{A}u = -\Delta u$. The case where $\alpha = 0$, $\beta > 0$, and $\mathcal{A} = -\Delta$ corresponds to a model of ionized atmosphere.

Questions related to the longtime behavior of the solutions for the PDE system have excited considerable attention in recent years, and the class introduced in (1) assume a variety of models which have been investigated in some limit cases $\alpha = 0$ and $\alpha = 1$.

In this paper, we study the existence and the asymptotic behavior of the solutions for system (1) when $\alpha \in]0, 1[$, $\beta = 0$ and the kernel g decays exponentially as $t \to \infty$.

Exponential decay of the energy was obtained when $\alpha = 1$, $\beta = 0$ and the memory kernel decaying exponentially. Concerning these viscoelastic models, we have the results due to Fabrizio and Lazzari [3]. Under assumptions that the relaxation function satisfies thermodynamic restrictions, they prove existence, uniqueness and asymptotic stability for linear viscoelasticity, by using the theory of contraction semigroups in Hilbert spaces and the theorem of Datko.

In [6], Muñoz Rivera considers the equations for linear isotropic homogeneous viscoelastic solids of integral type that occupy a bounded domain $\Omega \subset \mathbb{R}^n$, with zero boundary and history data and vanishing body forces. Exponential decay of all second-order derivatives in L^2 is proved if the memory kernel agrees and satisfies differential inequalities that imply their exponential decay. This is done by deriving suitable differential relations between several skillfully chosen integral functionals.

For the equations of linear one-dimensional viscoelasticity,

$$u_{tt} - k_0 u_{xx} + k * u_{xx} = 0 \quad \text{in } [0, \pi] \times \mathbb{R}^+,$$
$$u(0, t) = u(\pi, t) = 0 \quad \text{in } \mathbb{R}^+,$$
$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) \quad \text{in } [0, \pi],$$
$$u(x, 0) - u(x, -s) = w_0(x) \quad \text{in } [0, \pi] \times \mathbb{R}^+,$$

Liu and Zheng [5] show the exponential decay of the energy if

$$k \geqslant 0, \quad k_0 - \int_0^{\infty} k(s)\, ds > 0, \quad k' + \delta k \leqslant 0, \quad \delta > 0.$$

The proof uses a semigroup formulation in a history space and characterizations of exponentially stable semigroups in Hilbert space.

In [4], Giorgi et al. consider a semilinear partial differential equation of hyperbolic type with a convolution term describing simple viscoelastic materials with fading memory, namely

$$u_{tt} - k(0)\Delta u - k' * \Delta u + h(u) = f \quad \text{in } \Omega \times \mathbb{R}^+,$$

$$u(x,t) = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+,$$
$$u(x,t) = u_0(x,t) \quad \text{in } \Omega \times \mathbb{R}^+,$$

with $k(0)$, $k(\infty) > 0$ and k decreasing. They study the asymptotic behavior of solutions and, in the autonomous case, obtain the existence of global attractors in Hilbert space.

Concerning the case $\alpha = 0$, we recall the model can represent a ionized atmosphere, if $\beta > 0$, $\mathcal{A} = -\Delta$. When the memory kernel decays exponentially, we point out a result of Muñoz Rivera et al. [7]. They show that the dissipation produced by conductivity kernel alone is not strong enough to produce an exponential decay of the solution.

The attention, here, is devoted to the interpolating cases $\alpha \in]0,1[$, whether the dissipation given by the memory kernel g is strong enough to produce a uniform rate of decay of the solution to system $u \in \mathcal{D}(\mathcal{A})$ satisfying

$$\begin{aligned} &u_{tt}(t) + \mathcal{A}u(t) - (g * \mathcal{A}^\alpha u)(t) = 0 \quad \text{in } L^2(0,T;\mathcal{H}), \\ &u(0) = u_0, \quad u_t(0) = u_1 \quad \text{in } \mathcal{H}. \end{aligned} \tag{2}$$

If so, what type of rate of decay can we expect? Our main result is that the solution for system (2) decays polynomially as $t \to \infty$, even if the memory kernel decays exponentially. We use sharp energy estimates method, looking for appropriate multipliers and Liapunov functionals.

Finally, we point out that a model equation like (2) may arise in the case of a generalized Kirchhoff viscoelastic bar, where a bending-moment relation with memory is considered. These assumptions lead to an equation of the form

$$u_{tt}(t) - \gamma_0 \Delta u_{tt}(t) + \gamma_1 \Delta^2 u(t) + h * \Delta^2 u(t) - \gamma_2 \Delta u + g * \Delta u(t) = 0,$$

where $\gamma_0, \gamma_1, \gamma_2$ are non-negative constants, h and g are memory kernels and u describes the transversal motion of the bar. The assumptions $\gamma_0 = 0$, $g \equiv 0$ has been considered in [2]; while the case $\gamma_0 = 0$, $h = g \equiv 0$ has been studied in [1]. We consider the case $\gamma_0 = \gamma_2 = 0$ and $h \equiv 0$, i.e., in view of our analysis, what we call a *weakly*, or non-exponentially, dissipative case.

The main result of this paper is to show that the dissipation given by the memory effect is not strong enough to produce exponential stability when $0 \leqslant \alpha < 1$. That is, we prove that the corresponding semigroup, associated to problem (2), does not decay exponentially when $0 \leqslant \alpha < 1$. On the other hand, we show that such dissipation is capable to produce polynomial decay in appropriate norms.

The paper is organized as follows. Section 2 is devoted to an overview on the functional setting and notation. The polynomial decay of the energy associated to model (2) is studied in Section 3. Finally, in Section 4 we show the non-exponential stability of the solutions.

2. Functional setting and notation

All along this paper we assume that the embedding $D(\mathcal{A}^r) \hookrightarrow D(\mathcal{A}^s)$ is compact for any $r > s \geqslant 0$. Let us denote the convolution product

$$(g * f)(t) := \int_0^t g(t-\tau)\, f(\tau)\, d\tau$$

and introduce the following notations:

$$(g \,\Box\, f)(t) := \int_0^t g(t-\tau)\big|f(t) - f(\tau)\big|^2\, d\tau,$$

$$(g \diamond f)(t) := \int_0^t g(t-\tau)\big[f(\tau) - f(t)\big]\, d\tau.$$

We recall some lemma, used in the sequel. The former two are a consequence of above definitions and of differentiation of the term $k \,\Box\, \varphi$.

Lemma 1. *For any function $k \in C(\mathbb{R})$ and any $\varphi \in W^{1,2}(0,T)$ we have that*

$$(k * \varphi)(t) = (k \diamond \varphi)(t) + \left[\int_0^t k(\tau)\, d\tau\right]\varphi(t).$$

Lemma 2. *For any function $k \in C^1(\mathbb{R})$ and any $\varphi \in W^{1,2}(0,T)$ we have that*

$$(k * \varphi)(t)\varphi_t(t) = -\frac{1}{2}k(t)\big|\varphi(t)\big|^2 + \frac{1}{2}(k' \,\Box\, \varphi)(t) - \frac{1}{2}\frac{d}{dt}\left\{(k \,\Box\, \varphi)(t) - \left[\int_0^t k(\tau)\, d\tau\right]\big|\varphi(t)\big|^2\right\}.$$

Lemma 3. *For any function $k \in C(\mathbb{R})$ and any $\varphi \in W^{1,2}(0,T)$ we have that*

$$\big|(k \diamond \varphi)(t)\big|^2 \leqslant \left[\int_0^T \big|k(\tau)\big|\, d\tau\right]\big(|k| \,\Box\, \varphi\big)(t).$$

Proof. By Hölder inequality, we have

$$\begin{aligned}
\big|(k \diamond \varphi)(t)\big|^2 &= \left|\int_0^t k(t-\tau)\big[\varphi(\tau) - \varphi(t)\big]\, d\tau\right|^2 \\
&\leqslant \left|\left[\int_0^T \big|k(t-\tau)\big|\, d\tau\right]^{1/2}\left[\int_0^T \big|k(t-\tau)\big|\big[\varphi(\tau) - \varphi(t)\big]^2\, d\tau\right]^{1/2}\right|^2 \\
&= \left[\int_0^T \big|k(\tau)\big|\, d\tau\right]\big(|k| \,\Box\, \varphi\big)(t). \qquad \Box
\end{aligned}$$

Lemma 4. *Let us denote by H a Hilbert space. For any function $k \in C(\mathbb{R})$, $k > 0$, any $\varphi \in W^{1,2}(0,T)$ and $f \in H$, for any $\varepsilon > 0$ there exists a positive constant C_ε such that*

$$\left|f(t)(k \diamond \varphi)(t)\right| \leqslant \varepsilon \left|f(t)\right|^2 + C_\varepsilon (k \,\Box\, \varphi)(t).$$

Proof. By Young inequality and Lemma 3, we have

$$\begin{aligned}\left|f(t)(k \diamond \varphi)(t)\right| &= \left|f(t)\right|\left|(k \diamond \varphi)(t)\right| \leqslant \left|f(t)\right|\left[\int_0^t k(\tau)\,d\tau\right]^{1/2}\left[(k \,\Box\, \varphi)(t)\right]^{1/2} \\ &\leqslant \varepsilon\left|f(t)\right|^2 + \underbrace{\frac{1}{4\varepsilon}\left[\int_0^T k(\tau)\,d\tau\right]}_{:=C_\varepsilon}(k \,\Box\, \varphi)(t)\end{aligned}$$

for any $\varepsilon > 0$. □

Here we assume $\mathcal{V} \hookrightarrow \mathcal{H} \equiv \mathcal{H}' \hookrightarrow \mathcal{V}'$ with continuous and dense injections. The brackets $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denote the duality between $\mathcal{V}'$ and $\mathcal{V}$.

We end this section establishing the existence, regularity and uniqueness result to problem (2).

Theorem 5. *Let us suppose that the initial data u_0 and u_1 belong to $D(\mathcal{A})$ and $D(\mathcal{A}^{1/2})$, respectively, and g is a C^1 function in $\mathbb{R}^+$. Then, there exists only one solution of* (2) *satisfying*

$$u \in C\big(0,T; D(\mathcal{A})\big) \cap C^1\big(0,T; D(\mathcal{A}^{1/2})\big) \cap C^2\big(0,T; D(\mathcal{H})\big).$$

Moreover, if the initial data satisfy

$$(u_0, u_1) \in D(\mathcal{A}^{s+1/2}) \times D(\mathcal{A}^s)$$

for $s \geqslant 0$, then the solution satisfies

$$u \in C\big(0,T; D(\mathcal{A}^{s+1/2})\big) \cap C^1\big(0,T; D(\mathcal{A}^s)\big) \cap C^2\big(0,T; D(\mathcal{A}^{s-1/2})\big).$$

Proof. The proof is based on the Galerkin method, and Gronwall inequality. Since the problem is linear and the solution is strong (in the sense that belongs to the domain operator $\mathcal{A}$ for any $t \in \mathbb{R}^+$), the uniqueness follows by standard methods. □

Remark 6. An alternative procedure to show the well posedness of problem (2) is to use the semigroup approach (see Section 4).

3. The polynomial decay

We consider the model problem

$$u_{tt}(t) + \mathcal{A}u(t) - (g * \mathcal{A}^\alpha u)(t) = 0 \quad \text{in } L^2(0,T;\mathcal{H}). \tag{3}$$

We assume that the kernel $g \in C^2 \cap W^{2,1}(\mathbb{R}^+)$ satisfies the following set of hypotheses:

$$g(s) > 0, \quad \forall s \in \mathbb{R}^+, \tag{4}$$

$$-c_0 g(s) \leqslant g'(s) \leqslant -c_1 g(s), \quad \forall s \in \mathbb{R}^+, \tag{5}$$

$$\left|g''(s)\right| \leqslant c_2 g(s), \quad \forall s \in \mathbb{R}^+, \tag{6}$$

$$1 - G(t) \geqslant c_3 > 0, \quad \forall t \in \mathbb{R}^+, \tag{7}$$

where c_i, $i = 0, 1, 2, 3$, are positive constants and $G(t) := \int_0^t g(s)\,ds$. We introduce the energy functionals

$$\mathcal{E}_1(t) = \frac{1}{2}\int_\Omega \left[\left|u_t(t)\right|^2 + \left|\mathcal{A}^{1/2}u(t)\right|^2 - G(t)\left|\mathcal{A}^{\alpha/2}u(t)\right|^2 + (g \,\Box\, \mathcal{A}^{\alpha/2}u)(t)\right] dx$$

$$\mathcal{E}_2(t) = \frac{1}{2}\int_\Omega \left[\left|\mathcal{A}^{(1-\alpha)/2}u_t(t)\right|^2 + \left|\mathcal{A}^{1-\alpha/2}u(t)\right|^2 - G(t)|\mathcal{A}^{\alpha/2}u|^2 + (g \,\Box\, \mathcal{A}^{1/2}u)(t)\right] dx.$$

Problem (3) is dissipative, as the following lemma shows.

Lemma 7. *Let us suppose that u is a solution of* (3)*, for initial data satisfying*

$$(u_0, u_1) \in D(\mathcal{A}) \times D(\mathcal{A}^{1/2}).$$

Then the energy identity can be written as

$$\frac{d}{dt}\mathcal{E}_1(t) = -\frac{1}{2}g(t)\int_\Omega \left|\mathcal{A}^{\alpha/2}u(t)\right|^2 dx + \frac{1}{2}\int_\Omega (g' \,\Box\, \mathcal{A}^{\alpha/2}u)(t)\,dx,$$

$$\frac{d}{dt}\mathcal{E}_2(t) = -\frac{1}{2}g(t)\int_\Omega \left|\mathcal{A}^{1/2}u(t)\right|^2 dx + \frac{1}{2}\int_\Omega (g' \,\Box\, \mathcal{A}^{1/2}u)(t)\,dx.$$

Proof. We consider Eq. (3) and test it with $u_t \in \mathcal{H}$, to get

$$\frac{1}{2}\frac{d}{dt}\left[\|u_t\|^2 + \|\mathcal{A}^{1/2}u\|^2\right] - \langle g * \mathcal{A}^{\alpha/2}u, \mathcal{A}^{\alpha/2}u_t\rangle = 0.$$

Now, use Lemma 2 with $\varphi = \mathcal{A}^{\alpha/2}u \in W^{1,2}(0, T; \mathcal{H})$ and substitute in the previous equality to obtain

$$\frac{1}{2}\frac{d}{dt}\left[\|u_t\|^2 + \|\mathcal{A}^{1/2}u\|^2 + g \,\Box\, \mathcal{A}^{\alpha/2}u - G(t)\|\mathcal{A}^{\alpha/2}u\|^2\right] = \frac{1}{2}g' \,\Box\, \mathcal{A}^{\alpha/2}u - \frac{1}{2}g(t)\|\mathcal{A}^{\alpha/2}u\|^2.$$

To complete the proof, we repeat the same steps with Eq. (3), tested with the function $\mathcal{A}^{1-\alpha}u_t$ and, then, applying the results of Lemma 2 using $\varphi = \mathcal{A}^{1/2}u \in W^{1,2}(0, T; \mathcal{H})$. $\square$

Let us introduce the functionals

$$\mathcal{F}(t) = \frac{1}{2}\big\|(g * \mathcal{A}^{\alpha/2}u)(t)\big\|^2 - \big\langle u_t(t), (g*u)_t(t)\big\rangle, \tag{8}$$

$$\mathcal{K}(t) = \big\langle u(t), u_t(t)\big\rangle. \tag{9}$$

The following lemmas resume the behavior of their derivatives.

Lemma 8. *With the same hypotheses as in Lemma* 7*, assume that* (5) *holds. Then, we have that*

$$\frac{d}{dt}\mathcal{F}(t) \leqslant -\frac{g(0)}{2}\big\|u_t(t)\big\|^2 + \frac{c_0^2}{g(0)}g^2(t)\big\|u(t)\big\|^2 + g(t)\big\|\mathcal{A}^{1/2}u(t)\big\|^2 + \frac{1}{g(0)}\left(\int_0^\infty \big|g''(s)\big|\,ds\right)\big(|g''| \,\square\, u\big)(t) + \big\langle \mathcal{A}u, (g' \diamond u)(t)\big\rangle.$$

Proof. We use the identities

$$(g*u)_t(t) = g(0)u(t) + (g' * u)(t) \tag{10}$$

$$= g(t)u(t) + (g' \diamond u)(t), \tag{11}$$

which are provided by easy calculations. Test Eq. (3) with $(g*u)_t \in \mathcal{V}$, and recall (10) and (11). Then,

$$\frac{d}{dt}\big\langle u_t, (g*u)_t\big\rangle = \big\langle\!\big\langle -\mathcal{A}u + g * \mathcal{A}^\alpha u, (g*u)_t\big\rangle\!\big\rangle + \big\langle g(0)u_t + g'u + g'' \diamond u, u_t\big\rangle.$$

By application of (11) and integrating by parts, we have

$$\big\langle\!\big\langle -\mathcal{A}u + g * \mathcal{A}^\alpha u, (g*u)_t\big\rangle\!\big\rangle = -\big\langle\!\big\langle \mathcal{A}u, gu + g' \diamond u\big\rangle\!\big\rangle + \frac{1}{2}\frac{d}{dt}\|g * \mathcal{A}^{\alpha/2}u\|^2.$$

Collecting the two previous equality, we have

$$\frac{d}{dt}\left[\frac{1}{2}\|g * \mathcal{A}^{\alpha/2}u\|^2 - \big\langle u_t, (g*u)_t\big\rangle\right] = \langle\!\langle \mathcal{A}u, gu + g' \diamond u\rangle\!\rangle - g(0)\|u_t\|^2 - \langle g'u + g'' \diamond u, u_t\rangle.$$

In view of (5), and using Young inequality and Lemma 4, we have

$$-\langle g'u + g'' \diamond u, u_t\rangle \leqslant c_0 g\langle u_t, u\rangle + \big|\langle g'' \diamond u, u_t\rangle\big| \leqslant \frac{g(0)}{2} + \frac{c_0^2 g^2}{g(0)}\|u\|^2 + \frac{1}{g(0)}\left(\int_0^\infty \big|g''(s)\big|\,ds\right)\big(|g''| \,\square\, u\big),$$

which completes the proof. □

Lemma 9. *With the same hypotheses as in Lemma* 7*, assume that* (7) *holds. Then, we have*

$$\frac{d}{dt}\mathcal{K}(t) \leqslant \|u_t\|^2 - \frac{c_3}{2}\|\mathcal{A}^{1/2}u\|^2 + \frac{1}{2c_3}\int_0^\infty g(s)\,ds\,(g \,\square\, \mathcal{A}^{\alpha/2}u).$$

Proof. We test Eq. (3), with u in $\mathcal{H}$ and Lemma 2 with $\phi = \mathcal{A}^\alpha u$, then

$$\begin{aligned}\frac{d}{dt}\langle u, u_t\rangle &= \|u_t\|^2 - \|\mathcal{A}^{1/2}u\|^2 + \big\langle u, g \diamond \mathcal{A}^\alpha u + G(t)\mathcal{A}^\alpha u\big\rangle \\ &= \|u_t\|^2 - \|\mathcal{A}^{1/2}u\|^2 + G(t)\|\mathcal{A}^{\alpha/2}u\|^2 + \langle \mathcal{A}^{\alpha/2}u, g \diamond \mathcal{A}^{\alpha/2}u\rangle \\ &\leqslant \|u_t\|^2 - \big(1 - G(t)\big)\|\mathcal{A}^{1/2}u\|^2 + \langle \mathcal{A}^{\alpha/2}u, g \diamond \mathcal{A}^{\alpha/2}u\rangle,\end{aligned}$$

indeed, $\|\mathcal{A}^\alpha \phi\| \leqslant \|\mathcal{A}\phi\|$ for every $\phi \in \mathcal{D}(\mathcal{A})$. Exploiting (7) and Lemma 4, we have

$$\frac{d}{dt}\langle u, u_t\rangle \leqslant \|u_t\|^2 - c_3\|\mathcal{A}^{1/2}u\|^2 + \frac{c_3}{2}\|\mathcal{A}^{\alpha/2}u\|^2 + \frac{1}{2c_3}\left(\int_0^\infty g(s)\,ds\right)(g \,\square\, \mathcal{A}^{\alpha/2}u)$$

which concludes the proof. □

Let us now introduce the functional

$$\mathcal{L}(t) = N\mathcal{E}_1(t) + N\mathcal{E}_2(t) + \mathcal{F}(t) + \frac{g(0)}{4}\mathcal{K}(t)$$

with $N > 0$ which is chosen suitably large. We can state the main result of this section. The following theorem provides the rate of decay of the energy of the solution to model problem (3).

Theorem 10. *Suppose that $g \in C^2 \cap W^{2,1}(\mathbb{R}^+)$, and* (4)–(7) *hold, and that the initial data satisfy*

$$(u_0, u_1) \in \big[D(\mathcal{A}^{1-\alpha/2}) \times D(\mathcal{A}^{(1-\alpha)/2})\big] \cap \big[D(\mathcal{A}^{1/2}) \times \mathcal{H}\big].$$

Then, the solution of Eq. (3) *decays polynomially to zero. That is, there exists a positive constant C such that*

$$\mathcal{E}_1(t) \leqslant \frac{C[\mathcal{E}_1(0) + \mathcal{E}_2(0)]}{t}.$$

Proof. We will assume that the initial data satisfy

$$(u_0, u_1) \in D(\mathcal{A}) \times D(\mathcal{A}^{1/2}).$$

Our conclusion will follows using density arguments. In view of (7), we have that $\int_0^\infty g(s)\,ds < 1$. By (6), and adding the inequality in Lemma 8 to the one in Lemma 9 multiplied by $g(0)/4$, we have

$$\begin{aligned}\frac{d}{dt}\left[\mathcal{F}+\frac{g(0)}{4}\mathcal{K}\right] &\leqslant -\frac{g(0)}{4}\|u_t\|^2+\frac{c_0^2g^2}{g(0)}\|u\|^2+\frac{c_2}{g(0)}\big(|g''|\,\square\,u\big)+g\|\mathcal{A}^{1/2}u\|^2\\ &\quad+\langle\mathcal{A}^{1/2}u,g'\diamond\mathcal{A}^{1/2}u\rangle-\frac{c_3g(0)}{4}\|\mathcal{A}^{1/2}u\|^2+\frac{g(0)}{8c_3}(g\,\square\,\mathcal{A}^{\alpha/2}u)\\ &\leqslant -\frac{g(0)}{4}\|u_t\|^2+\frac{c_0^2g^2}{g(0)}\|u\|^2+\frac{c_2^2}{g(0)}(g\,\square\,u)\\ &\quad+g\|\mathcal{A}^{1/2}u\|^2+\frac{g(0)}{8c_3}(g\,\square\,\mathcal{A}^{\alpha/2}u)\\ &\quad-\frac{c_3g(0)}{8}\|\mathcal{A}^{1/2}u\|+\frac{2}{c_3g(0)}\int_0^\infty\big|g'(s)\big|\,ds(g\,\square\,\mathcal{A}^{1/2}u)\end{aligned}$$

with use of Lemma 4. Observing that $|g'(s)|\leqslant c_0g(s)$ for every $s\geqslant 0$, and that $c_3<1$, we get

$$\begin{aligned}\frac{d}{dt}\left[\mathcal{F}+\frac{g(0)}{4}\mathcal{K}\right] &\leqslant -\frac{c_3g(0)}{8}\big(\|u_t\|^2+\|\mathcal{A}^{1/2}u\|^2\big)+\frac{c_0^2g^2}{g(0)}\|u\|^2\\ &\quad+\frac{c_2^2}{g(0)}(g\,\square\,u)+g\|\mathcal{A}^{1/2}u\|^2\\ &\quad+\frac{2c_0}{c_3g(0)}(g\,\square\,\mathcal{A}^{1/2}u)+\frac{g(0)}{8c_3}(g\,\square\,\mathcal{A}^{\alpha/2}u).\end{aligned}$$

Now recall (5), and add to the previous the inequality in Lemma 7, multiplied by N. Exploiting Poincaré inequality, we get

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) &\leqslant -\frac{N}{2}g(t)\|\mathcal{A}^{\alpha/2}u\|^2-\left(\frac{N}{2}-c_0^2\right)g(t)\|\mathcal{A}^{1/2}u\|^2\\ &\quad-\left[\frac{Nc_1}{2}-\frac{g(0)}{2c_3}\right](g\,\square\,\mathcal{A}^{\alpha/2}u)-\left[\frac{Nc_1}{2}-\frac{2c_0}{c_3g(0)}-\frac{c_2^2}{g(0)}\right](g\,\square\,\mathcal{A}^{1/2}u)\\ &\quad-\frac{c_3g(0)}{8}\big(\|u_t\|^2+\|\mathcal{A}^{1/2}u\|^2\big).\end{aligned}$$

Recall the definition of $\mathcal{E}_1$. If N is chosen suitably large, we can find a positive constant γ_0 such that

$$\frac{d}{dt}\mathcal{L}(t)\leqslant-\gamma_0\mathcal{E}_1(t).$$

By an integration with respect to t, we get

$$\mathcal{L}(t)-\mathcal{L}(0)+\gamma_0\int_0^t\mathcal{E}_1(\tau)\,d\tau\leqslant 0.$$

If N is chosen sufficiently large, the functional $\mathcal{L}$ is positive definite for any $t>0$. Then, we have

$$\int_0^t \mathcal{E}_1(\tau)\,d\tau \leqslant \mathcal{L}(0), \quad \forall t > 0.$$

We recall that, from Lemma 7 and hypotheses (4) and (5), $(d/dt)\mathcal{E}_1(t) \leqslant 0$. Thus, we have that

$$\frac{d}{dt}\bigl[t\mathcal{E}_1(t)\bigr] = \mathcal{E}_1(t) + t\frac{d}{dt}\mathcal{E}_1(t) \leqslant \mathcal{E}_1(t), \quad \forall t > 0.$$

By an integration with respect to t, we find

$$t\mathcal{E}_1(t) \leqslant \int_0^t \mathcal{E}_1(\tau)\,d\tau \leqslant \mathcal{L}(0), \quad \forall t > 0,$$

and it follows that there exists a positive constant C such that

$$\mathcal{E}_1(t) \leqslant \frac{C[\mathcal{E}_1(0) + \mathcal{E}_2(0)]}{t}. \qquad \square$$

Remark 11. Note that the polynomial decay obtained in Theorem 10 is not in the same norm for the solution and the initial data. Then, it is not possible to get exponential decay from the estimates above.

4. Non-exponential stability

The result proved in Theorem 10 is sharp. Indeed, assuming that the kernel $g(s)$ decays exponentially, the solution to problem (3), with the specified initial and boundary conditions, cannot decays exponentially.

To represent the solution to problem (2) as a semigroup acting on the appropriate phase space, we introduce the function

$$\eta^t(s) = u(t) - u(t-s) \tag{12}$$

representing the *relative history* of u. Without assumption on the vanishing of initial past history, by (12), Eq. (3) can be rewritten as

$$\begin{cases} u_{tt} + \mathcal{A}u - G_\infty \mathcal{A}^\alpha u + \int_0^\infty g(s)\mathcal{A}^\alpha \eta(s)\,ds = 0, \\ \eta_t - u_t + \eta_s = 0, \end{cases} \tag{13}$$

where $G_\infty := \int_0^\infty g(\tau)\,d\tau$, and in view of (7) we recall that $G_\infty \leqslant 1 - c_3$.

Let $L^2_g(\mathbb{R}^+, \mathcal{D}(\mathcal{A}^{\alpha/2}))$ be the g-weighted L^2 spaces of functions on $\mathbb{R}^+$ with values in $\mathcal{D}(\mathcal{A}^{\alpha/2})$ (as defined in Section 2), endowed with the inner product

$$\langle \eta_1, \eta_2 \rangle_g := \int_0^\infty g(s)\bigl\langle \mathcal{A}^{\alpha/2}\eta_1(s), \mathcal{A}^{\alpha/2}\eta_2(s)\bigr\rangle\,ds.$$

We consider the Hilbert space

$$Z := \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{H} \times L^2_g\bigl(\mathbb{R}^+, \mathcal{D}(\mathcal{A}^{\alpha/2})\bigr)$$

and setting $v := u_t$, and

$$U(t) := \big[u(t), v(t), \eta(t)\big]^\top, \qquad U_0 := [u_0, v_0, \eta_0]^\top \in Z,$$

we can represent the solution to problem (13) as an abstract linear evolution equation in the Hilbert space Z of the form

$$\begin{cases} U_t(t) = AU(t), \\ U(0) = U_0. \end{cases} \tag{14}$$

The operator A is defined as

$$A\begin{bmatrix} u \\ v \\ \eta \end{bmatrix} = \begin{bmatrix} v \\ -\mathcal{A}u + G_\infty \mathcal{A}^\alpha u - \int_0^\infty g(s)\mathcal{A}^\alpha \eta(s)\,ds \\ v - \eta_s \end{bmatrix}$$

with domain

$$\mathcal{D}(A) := \Bigg\{ U \in Z\colon\ \mathcal{A}U \in Z,\ \int_0^\infty g(s)\mathcal{A}^\alpha \eta(s)\,ds \in \mathcal{H}, \\ \eta_s \in L^2_g\big(\mathbb{R}^+, \mathcal{D}(\mathcal{A}^{\alpha/2})\big),\ \eta(0) = 0 \Bigg\}.$$

We use a result of Prüss [8] which establishes that a C_0-semigroup e^{At} acting on a Hilbert space X is exponentially stable if and only if $i\mathbb{R} \subset \rho(A)$ and there exists $M \geqslant 1$ such that $\|(i\lambda I - A)^{-1}\| < M$, $\forall \lambda \in \mathbb{R}$. In order to apply this result, we consider the spectrum of the operator $\mathcal{A}$ with homogeneous Dirichlet boundary conditions, namely

$$\begin{cases} \mathcal{A}e_\nu = \lambda_\nu e_\nu & \text{in } \Omega, \\ e_\nu = 0 & \text{on } \partial\Omega, \end{cases} \qquad \text{and} \qquad \|e_\nu\|_{L^2(\Omega)} = 1, \quad \nu \geqslant 1, \tag{15}$$

where $(\lambda_\nu)_{\nu \geqslant 1}$ is an increasing sequence of positive eigenvalues of finite multiplicity, tending to infinity, and $(e_\nu)_{\nu \geqslant 1}$ is the corresponding sequence of normalized eigenfunctions.

Here we assume that

$$\lambda_1 > G_\infty \lambda_1^\alpha.$$

Theorem 12. *Let the kernel g decays exponentially. Then the semigroup $U(t)$, associated to problem* (14)*, and acting on the phase space Z, is not exponentially stable.*

Proof. Let $F = [F_1, F_2, F_3]^\top \in Z$ and consider the equation

$$(i\lambda I - A)U = F, \quad \lambda \in \mathbb{R}.$$

This equation reads

$$\begin{cases} i\lambda u - v = F_1, \\ i\lambda v + \mathcal{A}u - G_\infty \mathcal{A}^\alpha u + \int_0^\infty g(s)\mathcal{A}^\alpha \eta(s)\,ds = F_2, \\ i\lambda \eta - v + \eta_s = F_3. \end{cases} \tag{16}$$

Set $F_1 = 0$, $F_2 = 0$, and $F_3 = \lambda_\nu^{(-\alpha+1-\epsilon)/2} e^{-\lambda_\nu^{1-\epsilon} s} e_\nu$. We look for solutions of the form

$$u = pe_\nu, \qquad v = qe_\nu, \qquad \eta(s) = \varphi(s)e_\nu$$

with $p, q \in \mathbb{C}$ and $\varphi \in L_g^2(\mathbb{R}^+)$. Then, system (16) becomes

$$\begin{cases} i\lambda pe_\nu - qe_\nu = 0, \\ i\lambda qe_\nu + \lambda_\nu pe_\nu - G_\infty \lambda_\nu^\alpha pe_\nu + \int_0^\infty g(s)\lambda_\nu^\alpha \varphi(s)\,ds\,e_\nu = 0, \\ i\lambda\varphi(s)e_\nu - qe_\nu + \varphi_s(s)e_\nu = \lambda_\nu^{(-\alpha+1-\epsilon)/2} e^{-\lambda_\nu^{1-\epsilon} s} e_\nu. \end{cases} \tag{17}$$

From $(17)_1$ and $(17)_2$ we get

$$-\lambda^2 pe_\nu + \lambda_\nu pe_\nu - G_\infty \lambda_\nu^\alpha pe_\nu + \int_0^\infty g(s)\lambda_\nu^\alpha \varphi(s)\,ds\,e_\nu = 0,$$

and choosing $\lambda = \sqrt{\lambda_\nu}$, by $(17)_{1,2}$ we find

$$G_\infty p = \int_0^\infty g(s)\varphi(s)\,ds. \tag{18}$$

Solving the ordinary differential equation in $(17)_3$, and using $(17)_1$, we are yield to

$$\varphi(s) = Ce^{-i\sqrt{\lambda_\nu}s} + p + \frac{\lambda_\nu^{(-\alpha+1-\epsilon)/2}}{i\lambda_\nu^{1/2} - \lambda_\nu^{1-\epsilon}} e^{-\lambda_\nu^{1-\epsilon} s}. \tag{19}$$

By initial data, we have $\eta(0) = 0$. Then,

$$C = -p - \frac{\lambda_\nu^{(-\alpha+1-\epsilon)/2}}{i\lambda_\nu^{1/2} - \lambda_\nu^{1-\epsilon}}$$

and (19) becomes

$$\varphi(s) = \left(-p - \frac{\lambda_\nu^{(-\alpha+1-\epsilon)/2}}{i\lambda_\nu^{1/2} - \lambda_\nu^{1-\epsilon}}\right) e^{-i\sqrt{\lambda_\nu}s} + p + \frac{\lambda_\nu^{(-\alpha+1-\epsilon)/2}}{i\lambda_\nu^{1/2} - \lambda_\nu^{1-\epsilon}} e^{-\lambda_\nu^{1-\epsilon} s}. \tag{20}$$

From (18) and (20), setting

$$g(s) = e^{-\gamma s}, \quad \gamma \in \mathbb{R}^+,$$

we obtain

$$p = \frac{\lambda_\nu^{(-\alpha+1-\epsilon)/2}}{\gamma + \lambda_\nu^{1-\epsilon}}.$$

Hence, for any $\alpha \in (0,1)$ there exists $\epsilon \in (0,1)$, with $\epsilon > \alpha$ such that

$$p \approx C\lambda_\nu^{(-\alpha-1+\epsilon)/2} \quad \text{as } \lambda_\nu \to \infty$$

and, recalling that $u = pe_\nu$,

$$\|u\|_{\mathcal{D}(\mathcal{A}^{1/2})} \approx \lambda_\nu^{(\epsilon-\alpha)/2} \to \infty \quad \text{as } \lambda_\nu \to \infty.$$

Hence, the conclusion follows. □

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